eiSPLIT: an eigenvalue-method for Shear-Wave Splitting Analysis

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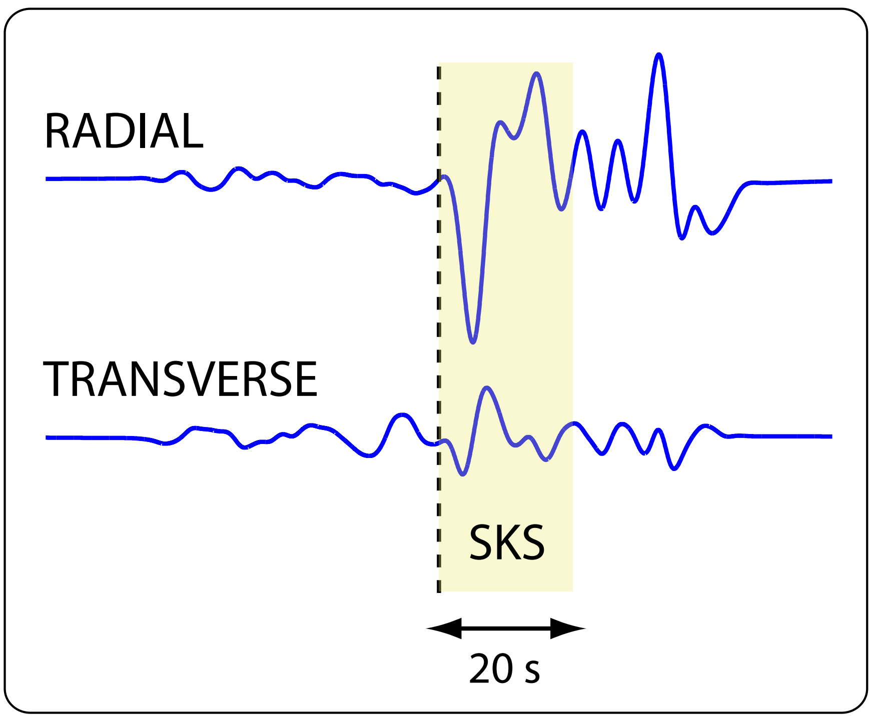

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I present here a suite of Matlab scripts aimed at conducting shear wave splitting measurements following the eigenvalue-minimization (EM) method of Silver and Chan [1991]. In this document I present the theory underlying the splitting of elastic shear waves and the EM method. Then I detail the use of each script with a demo dataset.

If you plan to use this code for research applications, please reference Olive et al. [2014].

1 Theoretical background

1.1 Elastic waves in an anisotropic medium

The propagation of seismic waves in a homogeneous elastic medium characterized by an elasticity tensor C_{ijkl} follows the acoustic wave equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl}\ \partial_j \partial_l u_k \tag{1}$$

where u_i is the displacement vector, ρ the medium density and the convention of repeated indexes is used. Solutions of (1) include plane waves of the form

$$u_i = A(\omega)\ p_i\ e^{-i\omega(t-b_j x_j)} \tag{2}$$

where $A(\omega)$ is the amplitude, and p, b and x are polarization, slowness[1] and position vectors, respectively. In this case, (1) may be rewritten as

$$\left[C_{ijkl} b_j b_l - \rho \delta_{ik}\right] p_k = 0 \tag{3}$$

which, after introducting the normalized slowness vector $b_i^* = c\ b_i$ and the Christoffel matrix $M_{ik} = \frac{1}{\rho} C_{ijkl} b_j^* b_l^*$, becomes

$$\left[M_{ik} - c^2 \delta_{ik}\right] p_k = 0 \tag{4}$$

Equation (4) describes an eigenvalue problem (Backus [1965]). For a weakly anisotropic medium, its first solution corresponds to P-waves, with a squared velocity α^2 corresponding to the maximum eigenvalue of M_{ik}, and polarization parallel or near-parallel to the slowness vector.

The other two solutions typically have a polarization orthogonal or quasi-orthogonal to b, and therefore describe elastic shear-waves. For isotropic media, these solutions both correspond to the same eigenvalue β^2, and describe S-waves. In the case of an anisotropic medium, M_{ik} may have two distinct eigenvalues defining the velocities of two quasi-S (split) waves. One can therefore define a coordinate system (A for anisotropic) along the ray path using the corresponding eigenvectors (f, s), where f and s describe the fastest and slowest components of the quasi-shear wave, respectively (Fig. 1b.).

[1] direction of propagation with norm equal to the inverse of velocity c

1.2 Shear wave splitting

1.2.1 Splitting operator

Shear wave splitting provides a unique insight into seismic anisotropy, provided a few simplifying assumptions are verified. Namely, the anisotropy is assumed to be weak and hexagonally symmetric about a vertical axis, i.e. the fast- and slow- directions are confined to the horizontal plane (i.e. transverse isotropy). Teleseismic shear waves emanating from distant sources are best suited to measure such anisotropy, as their ray path becomes close to vertical on the receiver side.

The two parameters of interest used to describe anisotropy are (1) the azimuth Φ of the fast polarization direction, and (2) the total time offset δt accumulated between the fast and slow components (Fig. 1a.).

A shear wave propagating vertically through a purely isotropic medium yields the following signal recorded at the surface:

$$\vec{u}_{\text{iso}} = A(\omega)\ \vec{p}\ e^{-i\omega t} \tag{5}$$

The effect of an anisotropic layer can be modelled by applying a splitting operator $\mathbb{S}$ to (5). This operator consists in projecting the initial polarization vector p (horizontal) on f and s and time-shifting each component by $\pm\delta t/2$, yielding

$$\vec{u}_{\text{split}} = \mathbb{S}(\vec{u}_{\text{iso}}) = A(\omega) e^{-i\omega t} \left[e^{-i\omega \frac{\delta t}{2}}\ (\vec{p}\cdot\vec{s})\ \vec{s} + e^{i\omega\frac{\delta t}{2}}\ (\vec{p}\cdot\vec{f})\ \vec{f} \right] \tag{6}$$

This process is illustrated in Fig. 1c.

1.2.2 Eigenvalue method

Let us consider a seismic station recording the east-west ($i = 1$) and north-south ($i = 2$) components of a split wave with fast azimuth Φ^* and delay time δt^*. An intuitive way to estimate these parameters is to apply an unsplitting operator $\mathbb{S}^{-1}(\delta t, \Phi)$ to the signal with numerous trial combinations of $(\delta t, \Phi)$, and look for the one that provides the best correlation between the unsplit components.

To do so, we apply the method described by Silver and Chan [1991]. For each pair of trial parameters $(\delta t, \Phi)$, we first calculate the cross-correlation matrix of the split wave in geographic (G) coordinates.

$$C^G_{ij}(\delta t) = \int_{-\infty}^{+\infty} u_i(t) u_j(t - \delta t)\ dt \tag{7}$$

We then rotate it into anisotropic (A) coordinates by applying the rotation matrix

$$R^{GA} = \begin{pmatrix} \sin\Phi & \cos\Phi \\ -\cos\Phi & \sin\Phi \end{pmatrix} \tag{8}$$

which yields

$$C^A_{ij}(\delta t, \phi) = R^{GA}_{ik}(\phi)\ C^G_{kl}(\delta t)\ R^{GA}_{lj}(\phi)^{-1} \tag{9}$$

As $(\delta t, \Phi)$ near the actual parameters $(\delta t^*, \Phi^*)$, C^A_{ij} resembles the cross-correlation matrix of an isotropic wave, which is by definition singular. However, the presence of noise

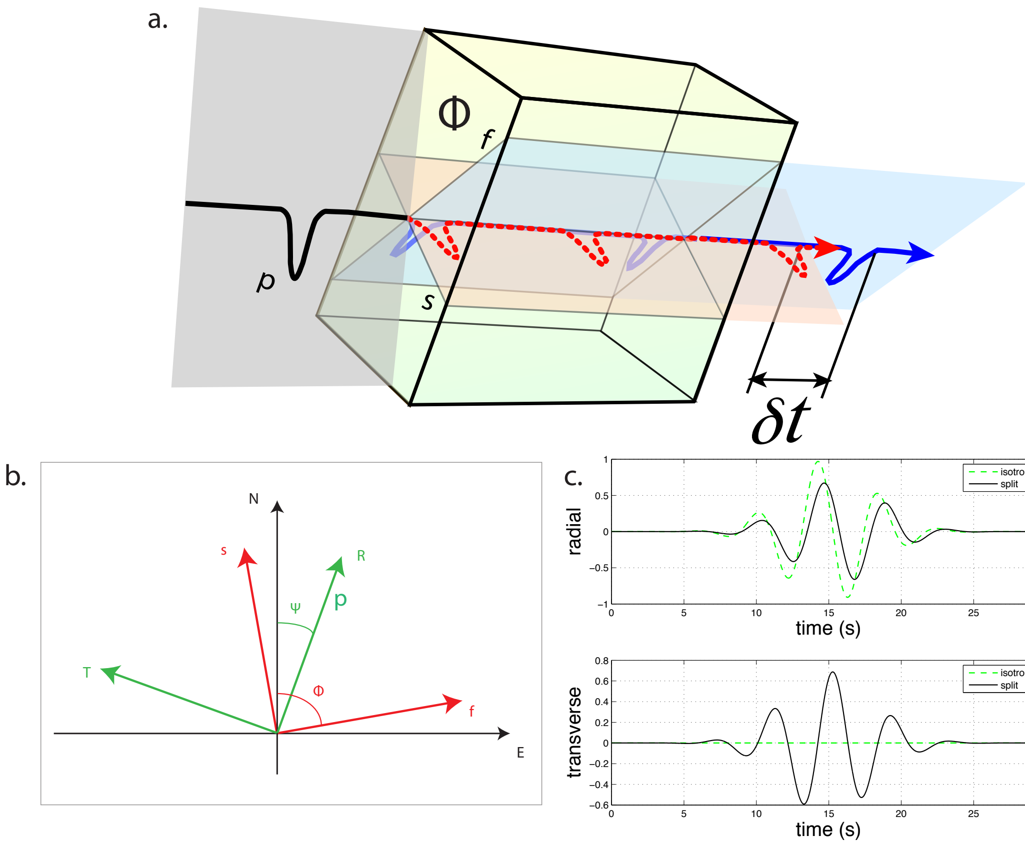

Figure 1: Illustration of the splitting process. **a.** A shear wave propagating through an isotropic medium enters an anisotropic region (green) characterized by a fast and a slow direction (blue and red planes) along which the initial pulse splits, accumulating a time offset δt. (Figure modified after E. Garnero and M. Crampin). **b.** Coordinate systems and notations used in this study: G (geographic) = east-west, A (anisotropy) = fast-slow and W (wave) = radial-transverse. Ψ and Φ denote the azimuth of polarization of the original isotropic shear wave and the azimuth of the fast splitting direction, respectively. **c.** Comparing an incident (isotropic) shear wave (dashed green line) and its splitted counterpart (black) in R-T coordinates, for $\Psi = 20°$, $\Psi = 80°$ and $\delta t = 1.3$ s.

in the signal will often prevent C_{ij}^A from being perfectly singular, i.e. from having a null-eigenvalue. We therefore look for the trial parameters that minimize the smallest eigenvalue λ_2 of C_{ij}^A, which is a good measure for its degree of singularity.

To find this minimum, we construct a map of λ_2 for each trial parameter (typically δt from 0 to 4 s and Φ from 0 to 180°) and contour a region of 95% confidence around the smallest λ_2.

1.2.3 Quality of measurements

The eigenvalue-minimization method has the advantage of being applicable to any shear wave phase (S, SS, SKS, ...). In practice, however, SKS and SKKS phases are especially well-suited for splitting measurements because the core-mantle conversion from P to S erases any effect of anisotropy on the source side, and produces a radially-polarized shear wave on the receiver side. Consequently, the unsplit (corrected) shear wave should have no energy on its tranverse component, providing a strong diagnostic tool for measurement quality.

Other indicators of a good measurement include: (1) a well defined pulse of energy on the radial and transverse components, (2) elliptic particle motion for uncorrected fast and slow components which becomes (3) a linear particle motion after time-shifting (correction), and (4) a well-defined, unique minimum on the λ_2 map. It is also recommended to make sure that the measurements are not sensitive to slight changes in the time (and frequency) window selected for the analysis.

Error bars are attributed to each measurement based on the method developped by Silver and Chan [1991]. A contour line of 95% confidence is defined on the λ_2 map based on the assumption that $\min(\lambda_2)$ is the sum-of-squares of a χ^2-distributed noise process. The confidence region therefore verifies

$$\frac{\lambda_2}{\min(\lambda_2)} \leq 1 + \frac{k}{\nu - k}\ f_{k,\nu-k}(1-\alpha) \tag{10}$$

with $\alpha = 0.05$, $k = 2$ and ν computed from waveform data following the appendix of Silver and Chan [1991]. The numerical procedure used to estimate ν in EiSPLIT is borrowed from the SplitLab suite (Wustefield et al. [2008]). f denotes the inverse of the F-distribution.

Certain cases of no-observable splitting are associated with a characteristic contour pattern that provides very little constraint on δt and two acceptable values of Φ, one of which is close to the wave back-azimuth, and the other offset by 90°. This is termed a null-measurement, and can mean one of the following three possibilities: $\delta t = 0$, $\vec{p} = \vec{f}$ ($\Phi =$ back-azimuth) or $\vec{p} = \vec{s}$ ($\Phi =$ back-azimuth $+ 90°$).

2 Running eiSPLIT

2.1 List of scripts and requirements

In order to run the eigenvalue-minimization code one must be in a directory containing:
(1) the main script eiSPLIT.m and the actual splitting routine split.m

1. EXTRACT EARTHQUAKE & STATION INFO + WAVEFORMS

extract waveforms, event time, station location, epicentral distance, depth, back-azimuth, sampling rate.

station info + waveform files (e.g. SAC and scripts to read them e.g. readsac.m)

2. LOCATE SKS WAVE

predict SKS arrival time and ray inclination at station.

Taup toolkit

3. SIGNAL PROCESSING

detrend waveforms, select a window around SKS, taper it, filter between [lf hf], rotate to radial-transverse coordinates, compute signal-to-noise ratio and ν parameter.

frequency bounds for filter, bpfilt.m, taper.m, getnu.m, geo2wave.m

Plot of N/E components of SKS
Plot of R/T components of SKS

4. ACTUAL ROTATION CORRELATION METHOD

visually select the SKS wave on the R/T plot, compute normalized λ_2 map and error contours.

split.m

Plot normalized λ_2 map
display results on screen

5. PRODUCE DIAGNOSTIC PLOTS

rotate into fast and slow coordinates based on the best splitting estimates, then time-shift for correction. Save workspace variables.

geo2ani.m, ttshift.m, ani2geo.m, wave2geo.m

Plot of split and unsplit fast and slow components
Plot of split and unsplit waves in 3 coordinate systems: fast vs. slow, radial vs. transverse, north vs. east.

Figure 2: Flowchart for main script eiSPLIT.m. Required inputs / accessory scripts for each step are written in green and outputs in red.

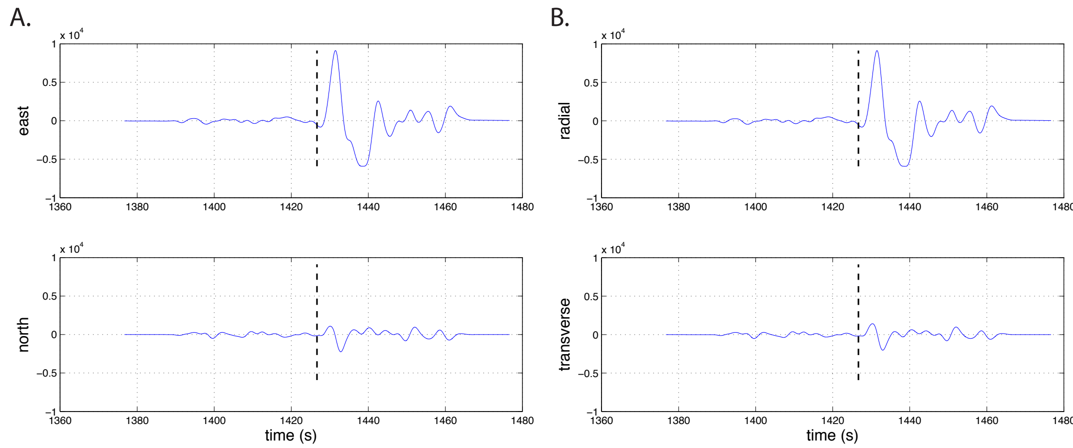

Figure 3: Filtered waveforms around the SKS arrival time (vertical dashed bar) in (A) N-E and (B) Radial-Transverse coordinates.

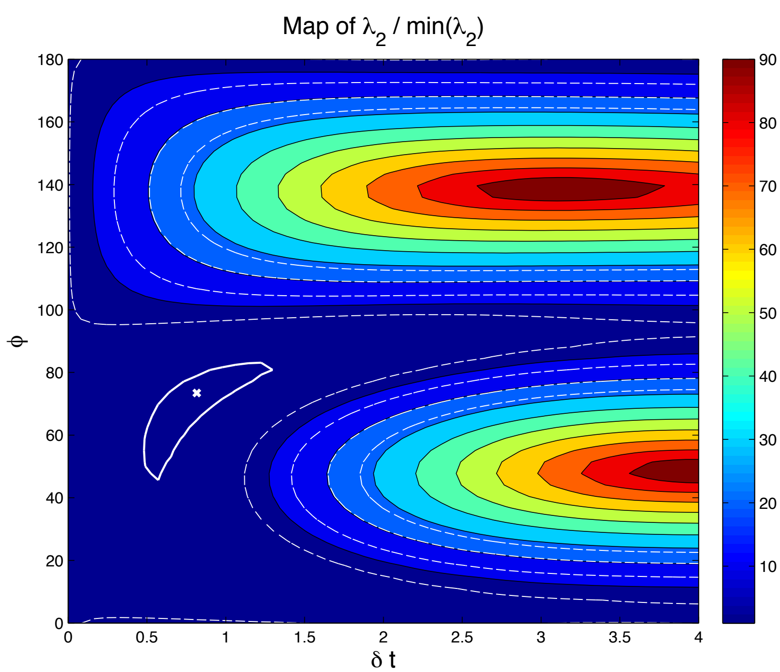

Figure 4: Map of normalized λ_2 and 95% confidence contour (thick white line) around the best estimate of δt and Φ.

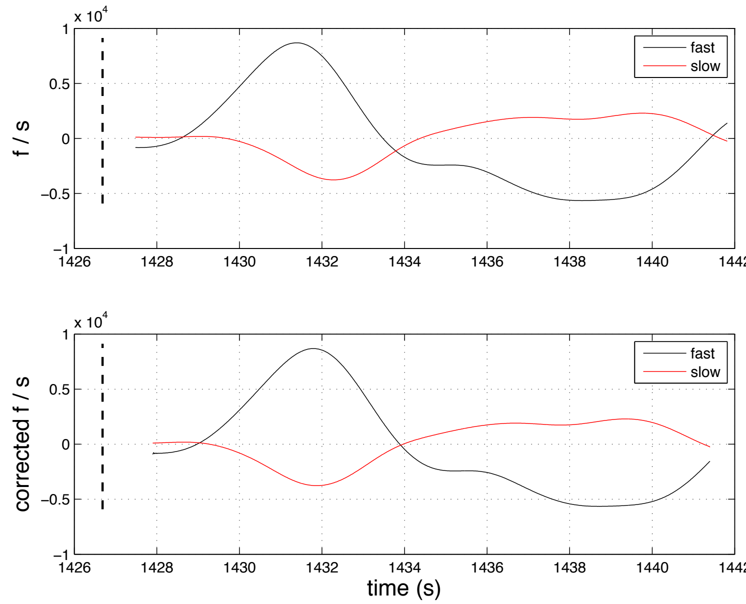

Figure 5: Fast and slow components of the split (top) and unsplit (bottom) SKS wave.

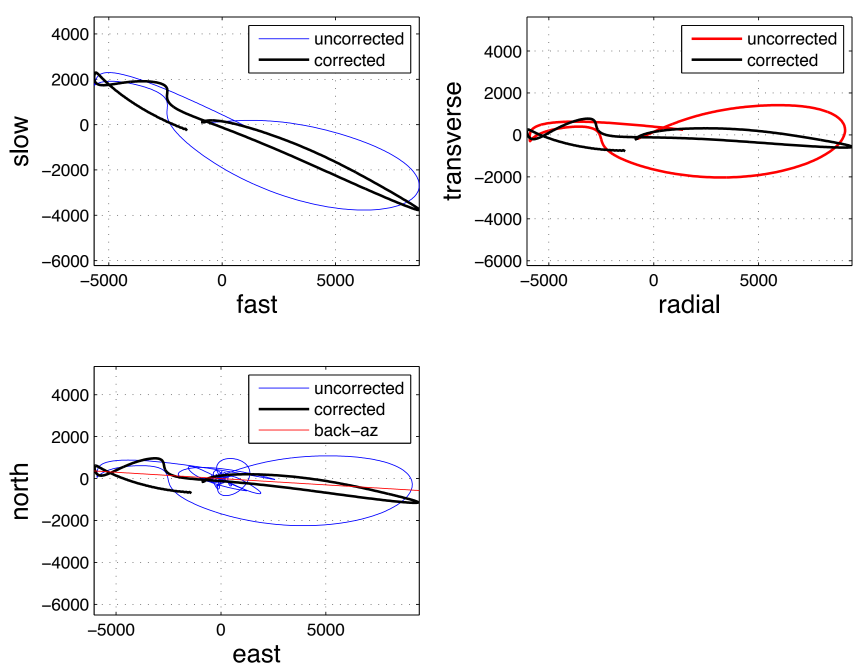

Figure 6: Split and unsplit SKS wave in fast vs. slow, radial vs. transverse and east vs. north coordinates.

(2) scripts performing changes in coordinate systems: geo2wave.m, geo2ani.m, wave2geo.m, ani2geo.m
(3) all scripts from-, or a link to the standard Taup toolklit (Crotwell et al. [1999])
(4) scripts for signal processing: bpfilt.m, ttshift.m, taper.m (written by S. Rondenay) and getnu.m
(5) scripts to extract waveform data, station and event info from relevant files (e.g. readsac, ...)

In its demo version, the code is presented with SAC files from a portable station in Northern Greece (Olive et al. [2014]) and the extraction of waveforms and station information is automated.

2.2 Step by step guide

In order to perform an example splitting measurement on this data, run eiSPLIT.m in a Matlab terminal. What happens next is summarized in a flowchart (Fig. 2).

Based on the event and station information, the Taup scripts compute a theoretical SKS arrival time. A $\sim$ 100 s window around that time is then selected, tapered and filtered between frequency bounds (lf and hf, to be specified, typically 0.01 to 0.4 Hz). The result is plotted in N-E and Radial-Transverse coordinates (Fig. 3).

Then, the user must select the actual SKS wave (typically $10-30$ s) by visual inspection on the Radial / Transverse vs. time plot (Fig. 3B). Using the matlab ginput tool, click on the start and end of the SKS wave, then hit enter. The normalized λ_2 map will then be computed and plotted (Fig. 4), and the best-fitting splitting estimates will be displayed on the screen along with error estimates and additional information:

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event
2007320
-----------
back-azimuth (deg)
  273.4474

-----------
Ray inclination (deg)
    8.8542

-----------
signal-to-noise ratio (dB)
   22.1251

-----------
best dt (s)
    0.7347

stdev (s)
```

```
   0.3673

best phi (deg)
   69.7959

stdev (deg)
   18.3673

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Based on the best Φ estimate, the SKS wave is rotated into fast vs. slow components. Then, based on the best δt estimate, these components are time-shifted to correct for the offset due to birefringence. This should yield very similar waveforms (possibly of opposite signs) on the corrected fast and slow components and provides a helpful diagnostic tool for measurement quality (Fig. 5).

Finally, the split and unsplit (corrected) SKS waves are plotted in fast vs. slow, radial vs. transverse and east vs. north coordinates (Fig. 6). The split wave should appear as an ellipse on all plots. The corrected wave should plot as a line (expressing the correlation between corrected fast and slow waves in Fig. 5) corresponding to the great axis of the ellipse. Corrected SKS waves should show very little to no energy on the corrected transverse component (the ellipse in a radial-transverse plot collapses to a flat line).

References

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